RINGS WITH INVOLUTION

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ABSTRACT
Structure theorems for rings R with involution whose symmetric elements satisfy a polynomial identity are obtained. In particular, it is shown that such rings satisfy polynomial identities.

1. **Introduction.** A ring R is said to be a ring with an involution if there exists a mapping $*: R \to R$ such that for every $a, b \in R$:

1) $a^{**} = a$
2) $(a+b)^* = b^* + a^*$
3) $(ab)^* = b^*a^*$.

The symmetric elements of R with respect to the involution (*) is the set $S = \{x \in R \mid x^* = x\}$; and the anti-symmetric elements is the set $K = \{x \in R \mid x^* = -x\}$.

The problem dealt with in this paper is the question: whether a ring R with an involution whose symmetric (or antisymmetric) elements satisfy a polynomial identity—necessarily also satisfies a polynomial identity. This has been shown to be true for simple rings as well as for semi-prime algebras of characteristic $\neq 2$; moreover, if the identity of S is of degree d then the degree of the identity of R is $\leqq 4d$. (Martindale [6], Herstein [2], [3]). In section 3 we prove this result for arbitrary rings (without the bound $4d$), and if R is a semi-prime algebra then the degree of the identity of R is $\leqq 2d$. By studying the nil subrings of R we obtain that $R/N_1(R)$ satisfy an identity of degree $\leqq 2d^2$, where $N_1(R)$ is the union of all nilpotent ideals of R. From which we conclude that every ring R of this type satisfies an identity.

2. **Nil subsets of R.** By a polynomial identity of S we mean a polynomial $p[x_1, \cdots, x_k]$ in non-commutative indeterminates x_i, for which at least one of the coefficients of a monomial of highest degree is 1, and which vanishes identically for all substitutions $x_i = s_i \in S$. The proof can be extended to more general identities, following [1], but for the sake of simplicity we restrict ourselves to the above polynomials. By a process of linearization we can get, and therefore we assume that $p[x]$ has the form:

$$(1.1) \qquad p[x_1, \cdots, x_2] = x_1 x_2 \cdots x_d + \sum d_{(i)} x_{i_1} \cdots x_{i_d}$$

Received December 28, 1967.

where $(i_1,\cdots,i_d)$ ranges over all permutations of $(1,2,\cdots,d)$ different from identity.

REMARK A. The property of the vanishing identity for the symmetric elements may not be invariant under homomorphisms which preserves the involution since the homomorphic image R/P of R may have more symmetric elements than those inherited fron R. This is not the case for algebras of over a field of characteristic $\neq 2$, since then if $r^* \equiv r(P)$ then $r^* - r = p \in P$ is an anti-symmetric element and hence $(r+p/2)^* = (r+p/2)$ with $r+p/2 \equiv r(\mathrm{mod}\, P)$. To overcome this difficulty we require less, and denote by S the set of all the symmetric elements of the form $\{a + a^*;\ ab^* + ba^*, \mid a, b \in R\}$, and $K = \{a - a^*; ab^* - ba^*\}$ and assume that this restricted set S satisfies the identity $p[x] = 0$. For this set S (and K) it is evident that the set $(S + P)/P$ is the restricted set of symmetric elements of R/P, and $p[x] = 0$ will vanish also in R/P. We refer henceforth only to the elements of this subset as to the symmetric elements of R.

We shall consider only identities of the symmetric elements S, but the proof for the anti-symmetric elements K is similar and suitable remarks of the changes required in the proofs will be enclosed.

We assume henceforth, that R is a ring with an involution * whose symmetric elements S satisfy the polynomial $p[x] = 0$ given in (1.1).

Our first result is:

LEMMA 1. *Let P be a two-sided ideal in R and let U be a subset of R such that $U^* = \{u^* \mid u \in U\} = U$ and $U^m \subseteq P$ then U^d generates a nilpotent ideal in R mod P.*

Proof. Following [1], we consider for $k > d$ the sets

$$U^{k-1}R,\ U^{k-1}RU,\ U^{k-2}RU,\ U^{k-2}RU^2,\cdots,$$

namely, the sets: $T_{2j-1} = U^{k-j}RU^{j-1}$, $T_{2j} = U^{k-j}RU^j$, $j = 1,2,\cdots,k$. Since $U^* = U$, it follows that $T^*_{2j-1} = T_{2(k-j+1)-1}$, $T^*_{2j} = T_{2(k-j)}$. Consider the symmetric elements(¹) $s_i = t_i + t_i^*$ where t_i is taken arbitrarily from T_i. First note that $t_it_j \in RU^kR$ if $i > j$.

For every permutation $(i_1,\cdots,i_d)$ of $(1,2,\cdots,d)$ we have:

$$\begin{aligned}(2.1)\qquad s_{i_1}\cdot s_{i_2}\cdots s_{i_d} &= (t_{i_1} + t^*_{i_1})\cdots(t_{i_d} + t^*_{i_d})\\ &= \sum_\lambda t_{i_{\lambda(1)}}\cdots t_{i_{\lambda(r)}}t^*_{i_{\lambda(r+1)}}\cdots t^*_{i_{\lambda(d)}} \mathrm{mod}(RU^kR)\end{aligned}$$

where the sum ranges over all permutations $(i_{\lambda(1)},\cdots,i_{\lambda(d)})$ of $(i_1,\cdots,i_d)$ and $r = 0,1,\cdots,d$. Indeed, since $i_\lambda \leqq d$ it follows that $t^*_{i_\lambda} \in T^*_{i_\lambda} = T_{j_\lambda}$ and $j_\lambda > d$, since we assumed $k > d$ and then for even $i_\lambda = 2j \leqq d$, $j_\lambda = 2(k-j) \geqq 2k-d > d$ and for odd $i_\lambda = 2j-1 \leqq d$, $j_\lambda = 2(k-j+1)-1 \geqq 2k-d > d$; this

(¹) For the anti-symmetric case we take $s_i = t_i - t_i^*$.

implies that $t_{i_\lambda}^* t_{i_\mu} \in RU^k R$ for $j_\lambda > d \geqq i_\mu$ and $t_{i_\lambda}^* \in T_{j_\lambda}$. Hence in the expansion of the left side of (2.1) we get $\bmod RU^kR$ only the terms on the right. Next, we show that:

$$s_{i_1} \cdots s_{i_d} U^h R \equiv t_{i_1} \cdots t_{i_d} U^h R \pmod{RU^k R} \tag{2.2}$$

where $h = d/2$ if d is even, and $h = (d+1)/2$ if d is odd. This follows from the fact that $t_i^* U^h R \subseteq U^j R U^{k-j+h} R$, where $i = 2j$ or $i = 2j - 1$; thus, we always have $k - j + h \geqq k$ since $i \leqq d$, and $h = d/2$ or $= (d+1)/2$. Finally, $t_{i_1} \cdots t_{i_d} U^h R \subseteq RU^k R$ if $(i_1, \cdots, i_d) \neq (1, 2, \cdots, d)$ since some pair $(i_\lambda, i_{\lambda+1})$ must satisfy $i_{\lambda+1} < i_\lambda$ and so $t_{i_\lambda} t_{i_{\lambda+1}} \in RU^k R$.

Hence substituting $x_i = s_i$ in $p[x]$ of (1.1) and multiplying on the right by $U^h R$ we get, in view of the preceding remarks, that $t_1 t_2 \cdots t_d U^h R = p[s_1, \cdots, s_d] U^h R \equiv 0(RU^k R)$. If we now let t_i range over the set T_i, we obtain $T_1 T_2 \cdots T_d U^h R = (U^{k-1}R)^d U^{r'+h} R \subseteq RU^k R$ where $2r' = d$ if d is even and $2(r'+1) - 1 = d$ if d is odd, and in both cases we conclude that $(RU^{k-1}R)^{d+1} \subseteq RU^k R$ if $k > d$.

Now let U^r be the minimal power of U which generates a nilpotent ideal in $R \bmod P$, then $r \leqq m$; from the last relation we conclude that $r \leqq d$, since if $r > d$ then $(RU^{r-1}R)^{d+1} \subseteq RU^r R$ and therefore $RU^{r-1}R$ will also be nilpotent $\bmod P$.

The preceding lemma will yield

THEOREM 2. *The nil radical $U(R)$ of R is equal to the lower radical $L(R)$ and $L(R)^d \subseteq N_1(R)$, where $N_1(R)$ is the union of all nilpotent ideals of R. Thus if $L(R) = 0$, then R has no nil ideals.*

Let $s \in S$ be a nil symmetric element, then by taking $U = (s)$ in the preceding lemma and $P = 0$, one obtains that s^d generates a nilpotent ideal in R, hence $s^d \in N_1(R) \subseteq L(R)$. Suppose $U(R)$ contains a non zero nilpotent symmetric element which $\notin L(R)$, then it must contain a symmetric $s \in U(R)$, $s \notin L(R)$ such that $s^2 \in L(R)$, then for every $x \in R$, $sx + x^*s \in U(R)$(2) is a nil and symmetric; hence Lemma 1 implies that $(sx + x^*s)^d$ generates a nilpotent ideal in R and it follows as before that $(sx + x^*s)^d \in L(R)$. Then $(sx)^{d+1} = (sx + x^*s)^d sx \in L(R)$, which means that if $(sR + L(R))/L(R) \neq 0$ it is a nil ring of bounded index, hence if $N/L(R)$ is the sum of all nilpotent ideals $T/L(R)$ in $[sR + L(R)]/L(R)$ then $(sR)^d \subseteq N$ ([1]). For every T, TsR will be a right nilpotent ideal mod $L(R)$ hence $TsR \subseteq L(R)$, so that $(sR)^{d+1} \subseteq L(R)$. But this implies that s generates a nilpotent ideal $\bmod L(R)$ which is impossible since $s \notin L(R)$, and $R/L(R)$ has no nilpotent ideals. Thus, we conclude that if $U(R) \neq L(R)$ then $S \cap U(R) \subseteq L(R)$. In this case, for $x \in U(R)$, $x + x^* \in U(R)$ since clearly $U(R)^* = U(R)$ and, therefore, $x + x^* \in L(R)$; and more generally $x^*t + t^*x \in L(R)$, for $x \in U(R)$, $t \in R$. Choose x such that $x \notin L(R)$, $x^2 \in L(R)$ then $x^*tx = (x^*t + t^*x)x \equiv 0(L(R))$ and $x \equiv -x^*(L(R))$

(2) For the non-symmetric case, we take $sx - x^*s$.

it follows that $(xR)^2 = -x^*RxR \equiv 0(L(R))$ which is impossible since $U(R)/L(R)$ has no nilpotent ideals. We conclude, therefore, that $U(R) = L(R)$.

Now let $u_1,\cdots,u_d \in L(R)$ and consider the finite set $U = (u_1,\cdots,u_d,u_1^*,\cdots,u_d^*)$ which satisfies $U^* = U$. By the local nilpotency of $L(R)$, it follows that $U^m = 0$ for some m. It follows now by the preceding lemma that U^d generates a nilpotent ideal, i.e., $U^d \subseteq N_1(R)$. In particular, $u_1u_2\cdots u_d \in N_1(R)$. This being true for all $u_i \in L(R)$, yields that $L(R)^d \subseteq N_1(R)$.

3. **Primitive images of R.** Let R be an arbitrary ring with an involution, and let P be a primitive ideal in R such that R/P is an irreducible ring of endomorphisms of a vector space V_D over a division ring D. Then:

LEMMA 3. *One of the following holds for R:*

1) *R/P has a minimal left ideal.*

2) *There is a finite dimensional D-subspace $W \subseteq V$, such that the left ideal $L = (0:W) = \{r \mid rW = 0\}$, satisfies $L^* \subseteq P$.*

3) *For every finite dimensional $U \subseteq V$, and every $v \notin U$, $[S \cap (0:U)]v \nsubseteq U + vD$.*

Proof. If (3) does not hold, then there exists a finite dimensional $U \subseteq V$, and $v \notin U$ such that $[S \cap (0:U)]v \subseteq U + vD$. Let $W = U + vD$ and $T = (0:U)$. By the density theorem it follows that $T \neq 0$ since T contains an element b such that $bU = 0$ and $bv \neq 0$. Furthermore: if $W = V$ then V is finite dimensional and hence R/P has a minimal left ideal, and if $V \neq W$, then for every $0 \neq a \in (0:W)$ and for every $r \in R$, $c = (ra)^*b + b^*(ra) \in (0:U)$(3) and c is symmetrical. Hence, it follows by our assumption that $(ra)^*bv = [(ra)^*b + b^*(ra)]v \in W$. Thus $a^*r^*bv \in W$, i.e., $a^*R(bv) \subseteq W$, and since $bv \neq 0$, we have $R(bv) = V$ so that $a^*V \subseteq W$. Now if $a^* \notin P$ for some $a \in (0:W)$ then it induces a linear transformation of finite rank, hence R/P has a minimal left ideal ([4], p. 75). Otherwise, $a^* \in P$ for every $a \in (0:W)$, and condition (2) is valid, as required.

LEMMA 4. *If R is a ring whose symmetric elements satisfy an identity, then every primitive image $\bar{R}$ of R has a minimal left ideal.*

Proof. Let $\bar{R} = R/P$ acting on V as in the preceding lemma, and we may assume that R acts on V. We have to consider only cases (2) and (3): in case (2), if $\bar{R}$ has no minimal left ideal, then necessarily $(V:D) = \infty$. Since $(W:D) < \infty$, one can choose $w_0, w_1, \cdots, w_d$ which are D-independent and $\notin W$. By the density theorem, we can choose $t_i \in L = (0:W)$ such that $t_iw_j = 0$ for $j \neq i$ and $t_iw_i = w_{i-1}$ for $i = 1,2,\cdots,d$, then since $t_i^* \in P$: $s_i = t_i + t_i^* \equiv t_i(\mathrm{mod}\,P)$(1), so that

$$0 = p[s_1,\cdots,s_d]w_d = p[t_1,\cdots,t_d]w_d$$
$$= t_1t_2\cdots t_dw_d = w_0$$

(3) Or, $(ra)^*b - b^*(ra)$ for the anti-symmetric case.

which is a contradiction, thus $(V:D) < \infty$ and consequently $\bar{R}$ has a minimal left ideal.

If case (3) holds, then we can choose $w \in V$, and $s_d, s_{d-1}, \cdots, s_1 \in S$ such that $w, s_d w, s_{d-1} s_d w, \cdots, s_k \cdot s_{k+1} \cdots s_d w$ are D-independent and $s_j(s_i s_{i+1} \cdots s_d w) = 0$ for $i < j+1$. This is carried out successively. First choose $w \neq 0$, then since $Sw \nsubseteq wD$ by (3) of the preceding lemma with $U = 0$, we choose $s_d w \notin wD$, and so independent of w. Suppose $s_d, \cdots, s_k$ have been chosen satisfying the above condition, let $U = wD + s_d wD + \cdots + s_{k+1} s_{k+2} \cdots s_d wD$, then since $[S \cap (0:U)](s_k s_{k+1} \cdots s_d w) \nsubseteq U + s_k s_{k+1} \cdots s_d wD$ we choose $s_{k-1} \in S \cap (0:U)$ such that $s_{k-1} s_k \cdots s_d w \notin U + s_k \cdots s_d wD$ and this satisfies our requirements. Next we substitute these s_i in $p[x]$ we get $0 = p[s_1 \cdots s_d]v = s_1 s_2 \cdots s_d v \neq 0$ which is a contradiction, and this completes the proof of the lemma.

We are now ready to prove the first main result:

THEOREM 5. *Let R be a ring with an involution whose symmetric elements S satisfy $p[x_1, \cdots, x_d] = 0$ then $R/L(R)$ satisfies an identity of degree $\leqq 2d$; and $R/N_1(R)$ satisfies an identity of degree $\leqq 2d^2$.*

Proof. Let V be an irreducible representation of R, and $P = \{r \mid rV = 0\}$ the primitive ideal of R. Consider first the case that $P^* \nsubseteq P$ (compare with [2] Lemma 25): the ring R/P contains the ideal $(P^* + P)/P$ and for each $t_i \in P^*$ we have $t_i^* \in P$ and therefore

$$0 = p[t_1 + t_1^*, \cdots, t_d + t_d^*] \equiv p[t_1, \cdots, t_d] \pmod{P},$$

that is $(P^* + P)/P$ satisfies an identity of degree d. Now a non-zero ideal in a primitive ring is primitive, and a primitive ring which satisfy an identity is a central simple algebra of dimension n $(\leqq [d/2])$. Let V_0 be an irreducible representation of $(P^* + P)/P$, then it is also an irreducible representation of R/P with the same centralizer, which implies that $R/P \cong \mathrm{Hom}_D(V_0, V_0)$ but the latter is isomorphic with $(P^* + P)/P$; so R/P is also a central simple algebra of dimension $n \leqq [d/2]$ over its center and in particular R/P satisfies the standard identity $S_{2d}[x] = 0$. (compane with [2], p. 228).

Next consider the case $P^* \subseteq P$, where then the involution of R induces an involution in R/P by setting $\bar{a}^* = \overline{a^*}$ for every $a \in R$. Since the symmetric elements of R satisfies the identity $p[x] = 0$, it will hold also in R/P (remark A), hence we may assume that R is primitive and it is an irreducible ring of endomorphisms of a space V_D. We shall prove first that $(V:D) \leqq d$, and this we do by induction on d:

It follows by Lemma 4 that R contains a minimal left ideal Re which can be taken to be V, and $D = eRe$. Let $e^*Re = M$, then $M \neq 0$ and both e, e^* are primitive idempotent hence M is a one-dimensional D-space. Indeed if v_1 $v_2 \in e^*Re$ and D-independent then there exist r such that $rv_1 = v_1, rv_2 = 0$ and clearly we can replace r by $e^*re^* \in D^*$ which is a division ring. Now for the division ring D^*, $rv_1 = v_1$ implies $r \neq 0$ but $rv_2 = 0$ yields that $r = 0$. Impossible! We define in $V = Re$ a bilinear map: $V \times V \to M$ by setting $(v_1, v_2) = v_1^* v_2$. Clearly this map is bilinear and satisfies $(rv_1, v_2) = (v_1, r^*v_2)$; $(v_1, v_2 d) = (v_1, v_2)d$, $d \in eRe$ and $(v_1 d, v_2) = d^*(v_1 v_2)$. It is hermitian in the sense that $(v_1, v_2)^* = (v_2, v_1)$, and regular for if $(v, V) = v^*V = 0$ then $v^* = 0$ and hence $v = 0$.

We prove that $(V:D) \leqq d$ by induction on d: consider first the case that there exist $v \in Re$ such that $(v,v) \neq 0$. Let $v^\perp = \{u \mid (u,v) = 0\}$, then since M_D is one dimensional it follows that $V = vD + v^\perp$. Consider the ring $R_0 = \{r \in R \mid rv = 0, rv^\perp \subseteq v^\perp\}$. Then $R_0^* \subseteq R_0$ and R_0 is an irreducible ring of transformations of v: Indeed, let $rv = 0$ then for every $u \in V$, $0 = (rv, u) = (v, r^*u)$ so that $r^*V \subseteq v^\perp$. Also for $r \in R_0$, $u \in v^\perp$, $0 = (v, ru) = (r^*v, u)$ since $rv^\perp \subseteq v^\perp$. Thus $(r^*v, v^\perp) = 0$ and $(r^*v, v) = (v, rv) = 0$, which implies by the linearity that $(r^*v, V) = 0$ hence the regularity implies that $r^*v = 0$, i.e.. $r^* \in R_0$. Next $v^\perp$ is clearly R_0-faithful but also it is an R_0-irreducible module: for if $u_1, u_2 \in v^\perp$ choose r such that $rv = 0$ and $ru_1 = u_2$. By the preceding proof it follows that $r^*V \in v^\perp$, and since R is a a primitive ring with a minimal left ideal we can choose r such that rV is one dimensional, so that $rV = u_2 D$. Indeed first choose arbitrary $r \in R$ by the density theorem, and then $e \in R$ such that $eu_2 = u_2$ and $eV = u_2 D$, then er is the required element. Thus $rv = 0$ and $rv^\perp \subseteq u_2 D \subseteq v^\perp$, i.e., $r \in R_0$. The symmetric elements of R_0 will satisfy an identity of degree $d-1$; for choose $x_i = s_i$, $i = 1, \cdots, d-1$ arbitrary symmetric elements in R_0 and $s_d = urv^* + vr^*u^*$ for arbitrary $r \in R$, $u \in v^\perp$ then $0 = p[s_1, \cdots, s_d]v = p_1[s_1 \cdots s_{d-1}]ur(v,v)$ since all $s_i v = 0$, $i \leqq d-1$ and $s_d v = ur(v,v)$ as $u^*v = (u,v) = 0$. This being true for all $r \in R$ implies $p_1[s_1 \cdots s_{d-1}]u = 0$ for all $u \in v^\perp$ hence $p_1[s_1 \cdots s_{d-1}] = 0$. Hence by induction $(v^\perp : D) \leqq d-1$, but then $V = vD + v^\perp$ yields that $(V:D) \leqq d$.

The second case where we have for all $v \in V$, $(v,v) = 0$ will follow similarly, but by passing to a ring R_0 whose symmetric elements satisfy an identity of degree $\leqq d-2$.

Here we choose two elements v_1, v_2 such that $(v_1, v_2) \neq 0$ and by assumption $(v_i, v_i) = 0$. Let $V_0 = v_1 D + v_2 D$ and consider $V_0^\perp = \{u \mid (v_i, u) = 0\}$. Then $V = V_0 + V_0^\perp$. The ring R_0 will now be defined as the set $\{r \mid rV_0^\perp \subseteq V_0^\perp, rV_0 = 0\}$. Again $R_0^* \subseteq R_0$, for if $r \in R_0$ for every $w \in V$, $0 = (rv_i, w) = (v_i, r^*w) = 0$ so that $r^*V \subseteq V_0^\perp$ and $0 = (v_i, ru) = (r^*v_i, u)$ for every $u \in V_0^\perp$. Thus $(r^*v_i, V) = 0$ which yields that $r^*V_0 = 0$, i.e., $r^* \in R_0$. Next V_0 is an irreducible R_0-module and R_0 acts faithfully on V_0 (i.e., R_0 primitive). The second assertion is evident, for the first we choose $u_1, u_2 \in V_0^\perp$ and $r \in R$ such that $rV_0 = 0$ $ru_1 = u_2$, and since R has a minimal left ideal we may take r to be a l.t. of rank 1 namely, $rV = u_2 D$, then $r \in R_0$ since $rV_0^\perp \subseteq u_2 D \subseteq V_0^\perp$ and $rV_0 = 0$.

Apply now the following substitution: for arbitrary $r_1, r_2 \in R$ and $u \in V_0^\perp$ we set $s_d = v_1 r_1 v_2^* + v_2 r_1^* v_1^*$ and $s_{d-1} = ur_2 v_2^* + v_2 r^* u^*$, $s_i \in R_0$ for $i = 1, 2, \cdots, d-2$.

$$= p[s_1,\cdots,s_d]v_1 = p_0[s_1,\cdots,s_{d-2}]s_{d-1}s_dv_1$$
$$= p_0[s_1\cdots s_{d-2}]ur_2(v_2v_1)r_1(v_2v_1).$$

Indeed, all $s_is_d = 0$ except $s_{d-1}s_d$ also all $s_iv_1 = 0$, $i \leqq d-2$, consequently, we have to consider only the monomials ending with $s_{d-1}s_dv_1$, which proves the preceding result since $(v_1v_1) = 0$, and $v_2^*v_1 = (v_2v_1)$ this being true for all $r_1, r_2 \in R$ and since $(v_1v_2) \neq 0$, yields that $p[s_1,\cdots,s_{d-2}]u = 0$ for every $u \in V_0^{\perp}$. Hence $p[s_1,\cdots,s_{d-2}] = 0$ in R_0. Thus, by induction: $(V_0{:}D) \leqq d-2$, as $V = V_0 + V_0^{\perp}$ it follows, $(V{:}D) \leqq d$. The case $d = 1,2$ are already included in the preceding proofs, and the induction is completed.

Thus we may assume that R acts on a finite dimensional vector space and so $R = D_k$. Let C be the center of R, then the involution of R induces an automorphism $c \to \bar{c}$ of degree $\leqq 2$ in C, and let C_0 be the invariant field. Let $F \subseteq D$ be a maximal commutative subfield of D. Consider the ring $R \otimes_{C_0} F = \tilde{R}$ as acting on V by setting $(r \otimes \alpha)v = rv\alpha$. The centralizer of $\tilde{R}$ in V is then F. Define in $\tilde{R}$ the involution $(\Sigma r_i \otimes d_i)^* = \Sigma r_i^* \otimes d_i$ then since we took the tensor product with respect to the invariant field C_0 this is well defined, since $(rc \otimes d_i)^* = (r \otimes cd_i)^*$; and since $p[x_1\cdots x_d]$ is multilinear, also the symmetric elements of $\tilde{R}$ satisfy $p[x] = 0$. Now let $P = \mathrm{Ker}[\tilde{R} \to \mathrm{Hom}_F(V,V)]$, and if $P^* \nsubseteq P$ it follows by the first case of primitive rings that $\tilde{R}/P$ satisfies the identity $S_{2d}[x] = 0$, and since $\tilde{R} \to R/P$ is an injection on the elements of R it follows that R satisfies this identity. (This will be the case $C_0 \neq C$.) If $P^* \subseteq P$, then by the second case of primitive ring it follows that $(V{:}F) \leqq d$ and, therefore, R is isomorphic with a subring of $\mathrm{Hom}_F(V,V)$ which satisfy $S_{2d}[x] = 0$. This concludes the proof for primitive rings, from which the semi-primitive case follows, since every primitive homomorphic image of satisfies $S_{2d}[x] = 0$ and R is a subdirect sum of its primitive images.

The semi-prime case follows now by embedding R in $R[t]$ the ring of polynomials in a commutative indeterminate t, and $R[t]$ is semi-primitive, since it follows by Theorem 2 that R has no nil ideals.

The final part of the proof of Theorem 5 is the obvious observation that $L(R)^* = L(R)$, for an ideal P is prime if and only if P^* is prime and $L(R) = \cap P$; and so $R/L(R)$ is semi-prime satisfying $p[x] = 0$ by remark A. Thus $R/L(R)$ satisfies $S_{2d}[x] = 0$, Theorem 2 implies that $L(R)^d \subseteq N_1(R)$ so that we conclude that $(S_{2d}[x])^d = 0$ in $R/N_1(R)$, and the proof of Theorem 5 is completed.

4. Arbitrary rings.

THEOREM 6. *If R is a ring with involution, such that its set S of symmetric elements satisfies a polynomial identity of degree d, then R satisfies an identity $S_{2d}[x]^m = 0$ for some m.*

Proof. Consider the complete product of the ring $\bar{R} = \Pi R_\alpha$ where α ranges over the set R^{2d} of all $2d$-tuple $(r_1,\cdots,r_{2d})$ of elements of R, and $R_\alpha = R$ for all α. $\bar{R}$ is also a ring with an involution by setting $f^*(\alpha) = [f(\alpha)]^*$ for all $\alpha \in \bar{R}$. Since for every component $p[x] = 0$ is satisfied by its symmetric elements—the same will hold in $\bar{R}$. So it follows that $S_{2d}[f_1, \cdots f_{2d}] \in L(\bar{R})$ for every $f_i \in \bar{R}$. Fixing $f_1, \cdots, f_{2d}$, the element $S_{2d}[f_i]$ belonging to the lower radical is nil—hence, $S_{2d}[f_i]^m = 0$. The way we choose the f_i is that f_i picks the i-th component of $\alpha = (r_1, \cdots, r_{2d})$, i.e., $f_i(\alpha) = r_i$. Thus $S_{2d}[f_i]^m(\alpha) = S_{2d}[r_1, \cdots, r_{2d}]^m = 0$. This being true for all α means that $S_{2d}[x_1, \cdots, x_{2d}]^m = 0$ holds identically in R. Q.E.D.

We conclude with

REMARK B. The integer m such that $S_{2d}[x]^m = 0$ holds in R is bounded by some integer depending on the identity $p[x]$ and not on R. If this were not the case one would have rings R_i satisfying $p=0$ and $S_{2d}[x]^{m_i} = 0$ with minimal m_i and $m_1 < m_2 < \cdots$. But then ΠR_i will satisfy $p = 0$ and will not satisfy any $S_{2d}^m = 0$. Contradiction.

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